Computation of the best Diophantine approximations by means of the generalized continued fraction

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Let in the real n-dimensional space $\mathbb{R}^n = \{X\}$ be given m real homogeneous forms $f_i(X)$, $i = 1, \dots, m$, $2 \leqslant m \leqslant n$. The convex hull of the set of points $G(X) = (|f_1(X)|, \dots, |f_m(X)|) \in \mathbb{R}^m_+$ for integer $X \in \mathbb{Z}^n$ in many cases is a convex polyhedral set. Its boundary for $||X|| <$ const can be computed by means of the standard program. Boundary points X, for which $G(X)$ are lying on the boundary, correspond to the best Diophantine approximations X for the given forms. Their computation gives the global generalization of the continued fraction. For $n = 3$ Euler, Jacobi, Dirichlet, Hermite, Poincaré, Hurwitz, Klein, Minkowski, Brun, Arnold and a lot of others tried to generalize the continued fraction, but without a succes.

Let $p(\xi)$ be an integer real irreducible in $\mathbb{Q}$ polynomial of the order n and λ be its root. The set of fundamental units of the ring $\mathbb{Z}[\lambda]$ can be computed using boundary points of some set of linear and quadratic forms, constructed by means of the roots of the polynomial $p(\xi)$. Up today such sets of fundamental units were computed only for $n = 2$ (using usual continued fraction) and for $n = 3$ (using the Voronoi algorithms). Each unit defines two automorphisms: (1) automorphism of boundary points in $\mathbb{R}^n$ and (2) automorphisms of their images in $\mathbb{R}^m_+$. In the logarithmic projection of $\mathbb{R}^m_+$ on $\mathbb{R}^{m-1}$ one can find the fundamental domain for the group of the automorphisms (2) [AB1].

Using these constructions, one can find integer solutions of Diophantine equations of a special form [AB2].

Our approach generalizes the continued fraction, gives the best Diophantine approximations, fundamental units of algebraic rings and solutions of some Diophantine equations for any n. Examples will be considered.

[AB1] A.D.Bruno, "Computation of the best Diophantine approximations and the fundamental units of the algebraic fields", Doklady Mathematics, 93:3, 243-247 (2016) DOI: 10.1134/S1064562416030017

[AB2] A.D.Bruno, "From Diophantine approximations to Diophantine equations", Preprint of KIAM, No. 1. Moscow (2016) (in Russian)

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